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Approximation of Wiener Integrals

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We present a new proof of Chorin's quadrature formula for Wiener integrals. Numerical experiments show that the rate of convergence may not be quadratic, if the functional is not twice differentiable. © 1987 Academic Press, Inc.

The purpose of this paper is to present a simple derivation of Chorin's [3] quadrature formula for Wiener Integrals.

The solution of the heat equation can be written as a Wiener integral. We can then compute the solution numerically by using Chorin's formula (see [3]). The technique has been extended to stochastic Wiener integrals by Blankenship and Baras [1]. Such integrals occur in the analysis of wave propagation in random media and in nonlinear filtering problems (see [1]). Chorin's formula may also find application in the calculation of the lowest eigenvalue of Schrödinger's equation (see Donsker and Kac [4]) and in the study of self avoiding random walk (see Ma [7, p. 400]).

Our presentation is based on two ideas. First, we use the Levy [6] interpolation formula and express the Brownian path as a sum of a piecewise linear function plus many small Brownian bridges. Here Chorin and Blankenship and Baras use piecewise constants plus many small Brownian paths. Second, we do not introduce extraneous variables only to eliminate them later by a change of variables.

The solution of the heat equation with a potential

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x, t) + V(x)\, u(x, t), \qquad -\infty < x < \infty,$$

$$u(x, 0) = f(x)$$

is given by the Feynman–Kac formula, (see, e.g., Freidlin [5, p. 120])

$$u(x, t) = \int_C f(x + z(t)) \exp\left\{\int_0^t V(x + z(s))\, ds\right\} dW_z.$$

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Here W_z is the Wiener measure in the space $C[0, t]$ of continuous functions z with $z(0)=0$. A special case of Chorin's formula gives

$$u(x, t)=\int_{\mathbb{R}^n} f(x+z_n) \exp\left\{\frac{t}{n} \sum_{i=1}^{n} V(x+z_i)\right\} d\mu+O(n^{-2}),$$

where

$$z_i=\frac{t}{\sqrt{n}}\left(u_1+\cdots+u_{i-1}+\frac{v}{\sqrt{2}}\right),$$

$$d\mu=(2\pi)^{-n/2} \exp\{-(u_1^2+\cdots+u_{n-1}^2+v^2)/2\}\, du_1 \cdots du_{n-1}\, dv.$$

This result holds if f and V are smooth and bounded. In practice the integral should be evaluated using Monte Carlo and variance reduction (see [2]). We will prove the result for $x=0$ and $t=1$, but replace the exponential function by a smooth function G and set $f \equiv 1$.

THEOREM (Chorin [3]). *Let G and V be four times continuously differentiable and assume that V and its derivatives are bounded. Then*

$$\int_C G\left(\int_0^1 V(x(t))\, dt\right) dW_x=\int_{\mathbb{R}^n} G\left(\frac{1}{n} \sum_{i=1}^{n} V(z_i)\right) d\mu+O(n^{-2}).$$

Warning. Here the Wiener measure satisfies $\int x^2(1)\, dW=1$. Chorin [3] takes $\int x^2(1)\, dW=\frac{1}{2}$ and this explains the difference in $d\mu$.

Remark. The smoothness is crucial. Numerical experiments show that quadratic convergence fails if $G(v)=v$ and $V(x)=|x|^\lambda$ with $0<\lambda<2$. The remainder in the theorem is less than $23KL^4/n^2$. This can be shown by replacing the estimates in our proof by explicit bounds. Here L is the supremum over all x of 1 and $|V^{(i)}(x)|$ for $i=0, 1, 2, 3, 4$, and K is the maximum of $|G^{(i)}(v)|$ for $i=0, 1, 2, 3, 4$, and $\inf_x V(x) \leqslant v \leqslant \sup_x V(x)$. There are other formulas that gives quadratic convergence. Vent-tsel', Gladyshev, and Mit'shteyn [8] has shown that if G is six times continuously differentiable and if V and its first four derivatives are continuous and bounded then

$$\int_C G\left(\int_0^1 V(x(t))\, dt\right) dW_x=\int_{\mathbb{R}^n} G\left(\frac{1}{n} \sum_{i=1}^{n} \frac{V(\hat{z}_{i-1})+V(\hat{z}_i)}{2}\right) d\hat{\mu}+O(n^{-2}),$$

where

$$\hat{z}_0=0, \qquad \hat{z}_i=\frac{1}{\sqrt{n}}(u_1+\cdots+u_i),$$

$$d\hat{\mu}=(2\pi)^{-n/2} \exp(-(u_1^2+\cdots+u_n^2)/2)\, du_1 \cdots du_n.$$

Proof. We begin with Levy's [6] representation of a Brownian path. Let $h=1/n$, $t_i=ih$, and $x_i=x(t_i)$. Let $t_{i-1}\leqslant t\leqslant t_i$. Then $x(t)=l_i+b_i$, where

$$l_i=x_0+(x_1-x_0)+\cdots+(x_{i-1}-x_{i-2})+\frac{t-t_{i-1}}{h}(x_i-x_{i-1}),$$

$$b_i=\frac{t_i-t}{h}(x(t)-x_{i-1})-\frac{t-t_{i-1}}{h}(x_i-x(t)).$$

The Brownian bridge b_i is independent of the linear interpolant l_i and

$$\int_C b_i\,dW=0,\qquad \int_C b_i^2\,dW=\frac{(t_i-t)(t-t_{i-1})}{h},$$
$$\int_C b_i^3\,dW=0,\qquad \int_C b_i^4\,dW=O(h^2). \tag{1}$$

We partition the inner integral as

$$\int_0^1 V(x(t))\,dt=\sum_{i=1}^n\int_{t_{i-1}}^{t_i}V(l_i)\,dt+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}[V(l_i+b_i)-V(l_i)]\,dt$$
$$=\sum_i a_i+\sum_i e_i=a+e.$$

Here and later i runs through 1, 2,..., n. By using Taylor's formula we get

$$I=\int_C G(a+e)\,dW=\int_C [G(a)+G'(a)\,e+\tfrac{1}{2}G''e^2]\,dW$$
$$=I_1+I_2+I_3.$$

We will show that $I_3=O(h^2)$. Since G'' is evaluated somewhere between a and $a+e$ it can be bounded independently of x. We observe next that $e^2=\sum_i e_i^2+\sum_{i\neq j}e_ie_j$. Since $e_i=\int V'b_i\,dt$ it follows from Cauchy–Schwarz inequality and (1) that

$$\int_C\sum_i e_i^2\,dW\leqslant\int_C\sum_i\left(\int_{t_{i-1}}^{t_i}\|V'\|\,|b_i|\,dt\right)^2 dW$$
$$\leqslant\sum_i\int_{t_{i-1}}^{t_i}\|V'\|^2\,dt\int_{t_{i-1}}^{t_i}dt\int_C b_i^2\,dW=O(h^2).$$

To evaluate the next term we expand $V(l_i+b_i)$ in a Taylor series. Since b_j is independent of l_i and b_i for $i\neq j$ and have zero mean we conclude that

$$\left|\int_C\sum_{i\neq j}e_ie_j\,dW\right|=\left|\sum_{i\neq j}\int_{t_{i-1}}^{t_i}ds\int_{t_{j-1}}^{t_j}dt\int_C(V'(l_i)\,b_i+\tfrac{1}{2}V''b_i^2)(V'(l_j)\,b_j+\tfrac{1}{2}V''b_j^2)\,dW\right|$$
$$\leqslant\tfrac{1}{4}\sum_{i\neq j}\|V''\|^2\int_{t_{i-1}}^{t_i}ds\int_C b_i^2\,dW\int_{t_{j-1}}^{t_j}dt\int_C b_j^2\,dW=O(h^2).$$

Thus $I_3 = O(h^2)$. We will now compute I_1. Since $x_0 = 0$ and the $(x_i - x_{i-1})$ are independent Gaussian variables with mean zero and variance h we have

$$I_1 = \int_{\mathbb{R}^n} G\left(\sum_i \int_{t_{i-1}}^{t_i} V(\lambda_i)\, dt\right) d\mu', \tag{2}$$

$$\lambda_i = u_1 + \cdots + u_{i-1} + \frac{t - t_{i-1}}{h} u_i,$$

$$d\mu' = (2\pi h)^{-n/2} \exp(-(u_1^2 + \cdots + u_n^2)/2h)\, du_1 \cdots du_n.$$

We denote the argument of G in (2) by α. To compute I_2 we expand $V(l_i + b_i)$ in a Taylor series. Since G' and V', V'', V''' do not depend on b_i and G' and V'''' are bounded it follows from (1) that

$$\begin{aligned} I_2 &= \int_C G'(a) \sum_i \int_{t_{i-1}}^{t_i} \left[V'(l_i)\, b_i + \frac{1}{2} V''(l_i)\, b_i^2 + \frac{1}{6} V'''(l_i)\, b_i^3 + \frac{1}{24} V''''b_i^4\right] dt\, dW \\ &= \sum_i \int_{t_{i-1}}^{t_i} dt \int_C G'(a) \frac{1}{2} V''(l_i)\, dW \int_C b_i^2\, dW + O(h^2) \qquad (3) \\ &= \int_{\mathbb{R}^n} G'(\alpha) \sum_i \int_{t_{i-1}}^{t_i} \frac{1}{2} V''(\lambda_i) \frac{(t_i - t)(t - t_{i-1})}{h}\, dt\, d\mu' + O(h^2). \end{aligned}$$

By combining (2) and (3) and changing the variable t to $\vartheta = (t - t_{i-1})/h$ we obtain

$$I = \int_{\mathbb{R}^n} \left[G(\alpha) + G'(\alpha)\, h \sum_{i=1}^n \int_0^1 \tfrac{1}{2} V''(u_1 + \cdots + u_{i-1} + \vartheta u_i)\, \vartheta(1 - \vartheta)\, h\, d\vartheta\right] d\mu' + O(h^2),$$
$$\alpha = h \sum_{i=1}^n \int_0^1 V(u_1 + \cdots + u_{i-1} + \vartheta u_i)\, d\vartheta.$$

Our next goal is to eliminate the ϑ variable inside V''. Set $\lambda_{i/i} = \lambda_i - \vartheta u_i$ and denote $V(\lambda_{i/i})$ by $V_{i/i}$. Then Taylor's formula yields

$$\int_0^1 \frac{1}{2} V''(\lambda_{i/i} + \vartheta u_i)\, \vartheta(1 - \vartheta)\, h\, d\vartheta = \frac{h}{12} V''_{i/i} + \frac{h}{24} V'''_{i/i} u_i + hO(u_i^2).$$

More generally we define $\lambda_{j/i}$ by setting u_i equal to zero in the expression for λ_j. Thus $\lambda_{j/i}$ equals $\lambda_j - u_i$ for $i < j$ and λ_j for $i > j$. We denote $V(\lambda_{j/i})$ by $V_{j/i}$. This notation reminds us that we have suppressed the u_i variable in V. By expanding V in a Taylor series we see that

$$\begin{aligned} \alpha &= h \sum_j \int_0^1 V_{j/i}\, d\vartheta + \left[\frac{h}{2} V'_{i/i} + h \sum_{j > i} \int_0^1 V'_{j/i}\, d\vartheta\right] u_i \\ &\quad + \left[\frac{h}{6} V''_{i/i} + \frac{h}{2} \sum_{j > i} \int_0^1 V''_{j/i} d\vartheta\right] u_i^2 + O(|u_i|^3) \qquad (4) \\ &= \alpha_i + p_i u_i + q_i u_i^2 + O(|u_i|^3). \end{aligned}$$

We observe next that

$$\int u_i\, d\mu' = 0, \qquad \int u_i^2\, d\mu' = h, \qquad \int u_i^3\, d\mu' = 0, \qquad \int |u_i|^m\, d\mu' = O(h^{m/2}). \tag{5}$$

Expanding G' around α_i and noting that α_i, p_i, and $\lambda_{i/i}$ do not depend on u_i, we conclude from (5) that

$$\int_{\mathbb{R}^n} G'(\alpha) \int_0^1 \tfrac{1}{2} V''(u_1 + \cdots + u_{i-1} + \vartheta u_i)\, \vartheta(1-\vartheta)\, h\, d\vartheta\, d\mu'$$
$$= \int_{\mathbb{R}^n} [G_i' + G_i'' p_i u_i + O(u_i^2)] \left[\frac{h}{12} V''_{i/i} + \frac{h}{24} V'''_{i/i} u_i + hO(u_i^2) \right] d\mu'$$
$$= \int_{\mathbb{R}^n} \frac{h}{12} G_i' V''_{i/i}\, d\mu' + O(h^2),$$

where $G_i = G(\alpha_i)$. By inserting this result in the previous expression for I we have

$$I = \int_{\mathbb{R}^{n+1}} \left[G(\alpha) + h \sum_{i=1}^n \frac{h}{12} G_i' V''_{i/i} \right] d\mu'' + O(h^2)$$
$$d\mu'' = (2\pi h)^{-(n+1)/2} \exp(-(u_1^2 + \cdots + u_n^2 + v^2)/2h)\, du_1 \cdots du_n\, dv.$$

Here we have introduced a dummy variable v and a new measure $d\mu''$. We shall replace $(h/12)\, V''$ by an approximation, which is good in the weak sense but bad in the absolute value sense. Expanding V around $\lambda_{i/i}$ and integrating with respect to ϑ we find that

$$\beta - \alpha = h \sum_i \int_0^1 \left[V\left(u_1 + \cdots + u_{i-1} + \frac{v}{\sqrt{2}}\right) - V(u_1 + \cdots + u_{i-1} + \vartheta u_i) \right] d\vartheta$$
$$= h \sum_i \left[V'\left(\frac{v}{\sqrt{2}} - \frac{u_i}{2}\right) + V''\left(\frac{v^2}{4} - \frac{u_i^2}{6}\right) + V'''\left(\frac{v^3}{12\sqrt{2}} - \frac{u_i^3}{24}\right) + O(v^4 + u_i^4) \right] \tag{6}$$

where V', V'', V''' are evaluated at $\lambda_{i/i} = u_1 + \cdots + u_{i-1}$. We remark that $\beta - \alpha$ is not a small quantity. The reason is that v is independent of the u_i's. But $\int |\beta - \alpha|$ is of order $\sqrt{h}$, while $\int \beta - \alpha$ is roughly $h \sum (h/12)\, V''$ because the integrals of the odd terms vanish. All integrations in the following are over $\mathbb{R}^{n+1}$. The domain of integration will therefore not be mentioned explicitly. We now expand G' around α_i, use (4), (5), and (6) and get

$$\int G'(\alpha)(\beta - \alpha)\, d\mu''$$
$$= h \sum_i \int \left[G' + G'' p_i u_i + \left(G'' q_i + \frac{1}{2} G''' p_i^2 \right) u_i^2 + O(|u_i|^3 + u_i^4) \right] (\beta - \alpha)\, d\mu''$$
$$= h \sum_i \int \left[G' V''_{i/i} \frac{h}{12} - G'' p_i V'_{i/i} \frac{h}{2} \right] d\mu'' + O(h^2),$$

where G', G'', G''' are evaluated at α_i. We may therefore replace $h \sum (h/12) V''$ by $\beta - \alpha$. More precisely, if we combine the last result with the expression for I we obtain

$$I = \int_{\mathbb{R}^{n+1}} \left[G(\alpha) + G'(\alpha)(\beta - \alpha) + h \sum_{i=1}^{n} \frac{h}{2} G_i'' p_i V'_{i/i} \right] d\mu'' + O(h^2).$$

The first two terms in this equation are the leading terms in the Taylor expansion of $G(\beta)$. Since β does not depend on u_n we can suppress this variable. After changing the variables u_i and v to $u_i'\sqrt{h}$ and $v'\sqrt{h}$ and dropping the primes we see that $\int G(\beta)\, d\mu''$ is Chorin's approximation to the Wiener integral. Chorin's quadrature formula is therefore first order accurate. To complete the proof we must show that the remainder is $O(h^2)$. Using the first five terms in the Taylor series for $G(\beta)$ we see that the remainder is

$$\int \left[h \sum_i \frac{h}{2} G_i'' p_i V'_{i/i} - \frac{1}{2} G''(\alpha)(\beta - \alpha)^2 - \frac{1}{6} G'''(\alpha)(\beta - \alpha)^3 - O(\beta - \alpha)^4 \right] d\mu''$$
$$= R_1 - R_2 - R_3 - R_4. \tag{7}$$

We shall show that R_1 and R_2 cancel, and that R_3 and R_4 are of order h^2. To estimate the last term is easy. It follows from (6) that $\beta - \alpha$ is $O(|v| + h \sum |u_i|)$. Since $(a+b)^4 \leqslant 8(a^4 + b^4)$ and $(h \sum |u_i|)^4 \leqslant h \sum u_i^4$ we find from (5) that

$$\int |\beta - \alpha|^4 \, d\mu'' \leqslant \text{const.} \int 8 \left(v^4 + h \sum_i u_i^4 \right) d\mu'' = O(h^2).$$

Thus $R_4 = O(h^2)$. We will now attack R_3. It follows from the derivation of (6) that

$$\beta - \alpha = cv - d + O\left(v^2 + h \sum u_i^2 \right),$$

where $c = (h/\sqrt{2}) \sum V'_{i/i}$ and $d = (h/2) \sum V'_{i/i} u_i$. But $cv - d$ is of order $(v^2 + h \sum u_i^2)^{1/2}$ and consequently

$$(\beta - \alpha)^3 = c^3 v^3 - 3c^2 v^2 d + 3cv\, d^2 - d^3 + O\left(v^4 + h \sum u_i^4 \right) + O\left(v^6 + h \sum u_i^6 \right).$$

We multiply both sides of this equation by $G'''(\alpha)$ and integrate with respect to $d\mu''$. The contribution from the last two terms will be $O(h^2 + h^3)$ since G''' is bounded. The terms with v^3 and v vanish after integration because G''', c, d are independent of v. To compute $\int G''' c^2 v^2 d$ we use Taylor's formula to suppress the u_i dependence in c and obtain

$$\int G'''(\alpha)\, c^2 v^2 d \, d\mu''$$
$$= \frac{h}{2} \sum_i \int \left[G_i''' + O(u_i) \right] \left[\frac{h}{\sqrt{2}} \sum_j V'_{j/ji} + O(u_i) \right]^2 h V'_{i/i} u_i \, d\mu'' = 0 + O(h^2).$$

We observe next that

$$\int G'''(\alpha)\, d^3\, d\mu'' = \frac{h^3}{8} \sum_{i,j,k} \int G'''(\alpha)\, V'_{i/i} V'_{j/j} V'_{k/k} u_i u_j u_k\, d\mu''.$$

If $i = j = k$ we use (4) and get

$$\int [G_i''' + O(u_i)](V'_{i/i})^3\, u_i^3\, d\mu'' = O(h^2).$$

Otherwise one of the indices must differ from the other two, say $i \neq j$ and $i \neq k$. Then

$$\int [G_i''' + O(u_i)]\; V'_{i/i}[V'_{j/ji} + O(u_i)][V'_{k/ki} + O(u_i)]\; u_i u_j u_k\, d\mu'' = O(h^2).$$

We can therefore conclude that $\int G''' d^3$ is of order h^2 and it follows that $R_3 = O(h^2)$. To compute R_2 in (7) we go back to (6) and write

$$\beta - \alpha = c_1 v - d_1 + c_2 v^2 - d_2 + O\left(|v|^3 + h \sum |u_i|^3 \right),$$

where $c_1 = c$ and $d_1 = d$. Consequently

$$(\beta - \alpha)^2 = c_1^2 v^2 + d_1^2 + 2(-c_1 v d_1 + c_1 c_2 v^3 - c_2 v^2 d_1 - c_1 v d_2 + d_1 d_2)$$
$$+ O\left(v^4 + h \sum u_i^4 \right) + O\left(v^6 + h \sum u_i^6 \right).$$

We multiply both sides of this equation by $\frac{1}{2} G''(\alpha)$ and integrate with respect to $d\mu''$. The terms with v and v^3 vanish after integration. The treatment of $c_2 v^2 d_1$ and $d_1 d_2$ is similar to the previous discussion of $c^2 v d$ and d^3, and the bounds are of order h^2. We observe next that

$$\int G''(\alpha)\, d_1^2 d\mu'' = \frac{h^2}{4} \sum_i \int G''(\alpha)(V'_{i/i})^2\, u_i^2\, d\mu''$$
$$+ \frac{h^2}{2} \sum_{i<j} \int G''(\alpha)\; V'_{i/i} V'_{j/j} u_i u_j\, d\mu''.$$

The sum of the diagonal terms is clearly $O(h^2)$. To estimate the second sum we use Taylor's formula to suppress the u_i and u_j dependence in $G''(\alpha)$ and $V'_{j/j}$ and get

$$\frac{h^2}{2} \sum_{i<j} \int [G_{ij}'' + G_{ij}'''(p u_i + q u_j) + O(u_i^2 + u_j^2)]$$
$$\times V'_{i/i}[V'_{j/ji} + r u_i + O(u_i^2)]\; u_i u_j\, d\mu'' = O(h^2),$$

because G''_{ij}, G'''_{ij}, p, q, r do not depend on u_i or u_j, and the terms with $u_i u_j$, $u_i^2 u_j$, $u_i u_j^2$ vanish after integration. This implies that $\int G'' d_1^2 = O(h^2)$. The main contribution to R_2 must therefore come from $\int G'' c_1^2 v^2$. Indeed, we have

$$\frac{1}{2}\int G''(\alpha)\, c_1^2 v^2\, d\mu'' = \frac{h^3}{4}\sum_i \int G''(\alpha)(V'_{i/i})^2\, d\mu'' + \frac{h^3}{2}\sum_{i<j}\int G''(\alpha)\, V'_{i/i} V'_{j/j}\, d\mu''.$$

The first sum is clearly $O(h^2)$. To evaluate the second sum we use Taylor's formula to suppress u_i and u_j in G'' and $V'_{j/j}$ and obtain

$$\begin{aligned} R_2 - O(h^2) &= \frac{h^3}{2}\sum_{i<j}\int [G''_{ij} + G'''_{ij}(pu_i + qu_j) + O(u_i^2 + u_j^2)] \\ &\quad \times V'_{i/i}[V'_{j/ji} + ru_i + O(u_i^2)]\, d\mu'' \\ &= \frac{h^3}{2}\sum_{i<j}\int G''_{ij} V'_{i/i} V'_{j/ji} d\mu'' + O(h^2). \end{aligned} \tag{8}$$

Finally we attack R_1 in (7). Using the definition of p_i we get

$$R_1 = \frac{h^2}{2}\sum_i \int G''_i p_i V'_{i/i} = \frac{h^3}{4}\sum_i \int G''_i (V'_{i/i})^2\, d\mu'' + \frac{h^3}{2}\sum_{i<j}\int G''_i V'_{i/i}\int_0^1 V'_{j/i} d\vartheta\, d\mu''.$$

The sum over the diagonal terms is $O(h^2)$. If we use Taylor's formula to suppress u_j in G''_i and $V'_{j/i}$ then the last sum equals

$$\begin{aligned} R_1 - O(h^2) &= \frac{h^3}{2}\sum_{i<j}\int [G''_{ij} + G'''_{ij} qu_j + O(u_j^2)] \\ &\quad \times V'_{i/i}[V'_{j/ij} + ru_j + O(u_j^2)]\, d\mu'' \\ &= \frac{h^3}{2}\sum_{i<j}\int G''_{ij} V'_{i/i} V'_{j/ji} d\mu'' + O(h^2). \end{aligned} \tag{9}$$

It follows from (8) and (9) that $R_1 - R_2 = O(h^2)$. The remainder (7) is therefore of order h^2. This completes the proof.

NUMERICAL EXPERIMENTS

In the proof G and V were smooth functions. We now set $G(v)=v$ and $V(x)=|x|^{\lambda}$ and consider the Wiener integral

$$J=\int_C \int_0^1 V(x(t))\,dt\,dW.$$

Since $x(t)$ is a gaussian random variable with mean zero and variance t we find after interchanging the order of integration and using a change of variables that

$$J=\int_0^1 dt\int_{-\infty}^{\infty} V(x)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx=\int_0^1 dt\int_{-\infty}^{\infty} V(\sqrt{t}\,y)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy. \tag{10}$$

The next step is to show that the quadrature formula by Vent-tsel', Gladyshev and Mit'shteyn [8] can be written as

$$\begin{aligned} J_1&=\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\frac{1}{2}\left[V\left(\frac{u_1+\cdots+u_{i-1}}{\sqrt{n}}\right)+V\left(\frac{u_1+\cdots+u_i}{\sqrt{n}}\right)\right]d\hat{\mu}\\ &=\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{1}{2}\left[V\left(\sqrt{\frac{i-1}{n}}\,y\right)+V\left(\sqrt{\frac{i}{n}}\,y\right)\right]\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy. \end{aligned}\tag{11}$$

Thus J_1 is the integral in (10) evaluated by the trapezoidal rule. We observe first that $x(j)-x(j-1)$ for $j=1,2,\ldots,$ are independent gaussian random variables with mean zero and variance one. This implies that

$$\begin{aligned} \int_C V\left(\frac{1}{\sqrt{n}}x(i)\right)dW&=\int_C V\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{i}[x(j)-x(j-1)]\right)dW\\ &=\int_{\mathbb{R}^n} V\left(\frac{u_1+\cdots+u_i}{\sqrt{n}}\right)d\hat{\mu}. \end{aligned}\tag{12}$$

Since the left-hand side of this equation is equal to

$$\int_{-\infty}^{\infty} V\left(\frac{x}{\sqrt{n}}\right)\frac{e^{-x^2/2i}}{\sqrt{2\pi i}}\,dx=\int_{-\infty}^{\infty} V\left(\sqrt{\frac{i}{n}}\,y\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

we have completed the proof of (11). For Chorin's quadrature formula one obtains

$$\begin{aligned} J_2&=\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^n} V\left(\frac{1}{\sqrt{n}}\left(u_1+\cdots+u_{i-1}+\frac{v}{\sqrt{2}}\right)\right)d\mu\\ &=\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} V\left(\sqrt{\frac{i-1/2}{n}}\,y\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy. \end{aligned}\tag{13}$$

This shows that J_2 is the integral in (10) evaluated by the midpoint rule. As in the derivation of (12) we see that

$$\int_{-\infty}^{\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}}\,dv \int_C V\left(\frac{1}{\sqrt{n}}\left(x(i-1)+\frac{v}{\sqrt{2}}\right)\right) dW$$
$$= \int_{\mathbb{R}^n} V\left(\frac{1}{\sqrt{n}}\left(u_1+\cdots+u_{i-1}+\frac{v}{\sqrt{2}}\right)\right) d\mu.$$

The left-hand side is also equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V\left(\frac{1}{\sqrt{n}}\left(x+\frac{v}{\sqrt{2}}\right)\right)\frac{e^{-x^2/2(i-1)}}{\sqrt{2\pi(i-1)}}\frac{e^{-v^2/2}}{\sqrt{2\pi}}\,dx\,dv$$
$$= \int_{-\infty}^{\infty} V\left(\sqrt{\frac{i-1/2}{n}}\,y\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \int_{-\infty}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz.$$

Here we have used the change of variables $y=(x+v/\sqrt{2})/\sqrt{i-1/2}$ and $z=-\alpha x+\beta v$ where $\alpha=1/\sqrt{(i-1)(2i-1)}$ and $\beta=\sqrt{2(i-1)/(2i-1)}$. Since the last integral is equal to one we have proved (13). We can now set $V(x)=|x|^{\lambda}$ in (10), (11) and (13) and get

$$J=C(1+\lambda/2)^{-1},$$
$$J_1=Cn^{-(1+\lambda/2)}\left(\sum_{i=1}^{n-1} i^{\lambda/2}+\tfrac{1}{2}n^{\lambda/2}\right),$$
$$J_2=Cn^{-(1+\lambda/2)}\sum_{i=1}^{n}\left(i-\tfrac{1}{2}\right)^{\lambda/2},$$

where $C=(2/\pi)^{1/2}\,2^{\lambda/2}\,\Gamma((1+\lambda)/2)$ and Γ denote the Gamma function. The problem has therefore been reduced to studying the accuracy of the trapezoidal rule and the midpoint rule applied to the integral $\int_0^1 t^{\lambda/2}\,dt$. If $\lambda=2$ then both quadrature formulas give the exact value.

Table I shows that the relative error in Chorin's formula is less than the relative

TABLE I

Relative Errors: $(J_i-J)/J$ for $\lambda=1$

n	Trapezoidal	Midpoint
1	$-0.250_{10}-0$	$0.607_{10}-1$
10	$-0.924_{10}-2$	$0.258_{10}-2$
100	$-0.306_{10}-3$	$0.882_{10}-4$
1000	$-0.980_{10}-5$	$0.286_{10}-5$

TABLE II

Asymptotic Error Constants: $n^{1+\lambda/2}$* Relative Error

n	$\lambda = .01$	$\lambda = .1$	$\lambda = 1.0$
1	$0.152_{10}-2$	$0.142_{10}-1$	$0.607_{10}-1$
10	$0.170_{10}-2$	$0.161_{10}-1$	$0.815_{10}-1$
100	$0.172_{10}-2$	$0.163_{10}-1$	$0.882_{10}-1$
1000	$0.172_{10}-2$	$0.163_{10}-1$	$0.903_{10}-1$

error in the formula of Vent-tsel', Gladyshev, and Mit'shteyn [8]. Note that the functional is Lipschitz continuous for $\lambda = 1$, and differentiable if $\lambda > 1$.

It follows from Table II that the rate of convergence for the midpoint rule is $n^{-(1+\lambda/2)}$ and the trapezoidal rule behaves similarly. This can also be proved analytically as long as $0 < \lambda < 2$. Thus the rate of convergence is not quadratic unless the functional is several times differentiable. Once is not enough.

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